Brownian Motion
Part III - An Introduction To Stochastic Calculus

Gary Schurman MBE, CFA

February 2012

In Parts I and II we developed the concept of Brownian motion, which is the source of randomness in equations that model the evolution of stock price over time. In this part of the series we will develop an equation for stock price using a branch of mathematics called Stochastic Calculus. Assume that we have a stock with the following parameters...

Stock price today	=	\$100	
Known rate of return on a risk-free asset	=	0.06	(compounded monthly)
Expected rate of return on a risky asset	=	0.18	(compounded monthly)
Risky asset annual return volatility	=	0.30	

Question: (1) What is expected stock price in six months time? (2) What is the probability that stock price will be greater than \$125.00?

Risk-Free Asset Prices in Discrete Time

An ordinary differential equation (ODE) defines the change in stock price to be a function of the current stock price, the periodic rate of return and the length of the time interval over which the change in stock price occurs. This ordinary differential equation is...

$$\Delta S_t = S_t \, \mu \, \Delta t \tag{1}$$

In the equation above S_t represents stock price at time t, μ represents the periodic rate of return and Δt represents the length of the time interval over which the change in stock price occurs. Note that μ and Δt are related in that μ is the rate of return for an entire period and Δt is the portion of that time period over which stock price changes. Also note that per the equation above stock price at any time is known with certainty and therefore the stock will earn the risk-free rate. An example application might be that we want to calculate the change in stock price over one month's time. In this example S_0 is \$100, μ is 6% and Δt is one month. Using Equation (1) above the change in stock price is...

$$\Delta S_t = S_t \, \mu \, \Delta t = \$100.00 \times 0.06 \times \frac{1}{12} = \$0.50 \tag{2}$$

Using the result of Equation (2) above stock price at the end of month one is...

$$S_{t+\Delta t} = S_t + \Delta S_t = \$100.00 + \$0.50 = \$100.50 \tag{3}$$

Stock price at the end of six months is...

$$\Delta S_{t+6\Delta t} = S_t + \sum_{i=1}^{6} \Delta S_{t+i-1} = \$103.04 \tag{4}$$

The table below presents the stock price calculation used in Equation (4) above...

Month	Beginning Price	Change in Price	Ending Price
1	100.00	0.5000	100.50
2	100.50	0.5025	101.00
3	101.00	0.5050	101.51
4	101.51	0.5075	102.02
5	102.02	0.5101	102.53
6	102.53	0.5126	103.04

We want to find an equation for stock price at any time t where $t > 0$. If S_0 is stock price at time zero, μ is the periodic rate of return, t is the number of time periods and each time period is divided into n discrete compounding periods then the equation for stock price at any time t using Equations (1) and (4) above is...

$$S_t = S_0 + \sum_{i=1}^{tn} \Delta S_{i-1} = S_0 + \sum_{i=1}^{tn} S_{i-1}\,\mu\,\Delta t \;\; ...where... \;\; \Delta t = \frac{1}{n} \tag{5}$$

Risk-Free Asset Prices in Continuous Time

With Stochastic Calculus we will be dealing with changes in asset prices over infinitesimally small time periods and therefore need continuous time versions of the discrete time equations above. The continuous time version of the ODE in Equation (1) above is...

$$\delta S_t = S_t\,\mu\,\delta t \tag{6}$$

Note that when converting from discrete time to continuous time we replace the ΔS_t and Δt in Equation (1) with δS_t and δt, respectively. Whereas [Δ] represents a change over a relatively large time interval [δ] represents a change over an infinitesimally small time interval (i.e. near zero). Also note that in discrete time the periodic rate of return is compounded in discrete time intervals whereas in continuous time the periodic rate of return is compounded continuously.

We want to find an equation for stock price in continous time using the ODE in Equation (6). This new equation will be the continous time version of Equation (5) above. When returns are compounded continuously the asset price path is exponential (i.e. non-linear). We can make the price path linear by using the log of stock price rather than the actual dollar price. Let's begin by defining the function $F(S_t)$ to be the log of stock price at any time t. Using a second-order Taylor Series Expansion the equation for the change in the log of stock price is...

$$\delta F(S_t) = \frac{\delta F(S_t)}{\delta S_t}\delta S_t + \frac{1}{2}\frac{\delta^2 F(S_t)}{\delta S_t^2}\delta S_t^2 \tag{7}$$

The first and second derivatives of our function $F(S_t)$ are...

$$F(S_t) = \ln S_t \;\; ...such\,that... \;\; \frac{\delta F(S_t)}{\delta S_t} = \frac{1}{S_t} \;\; ...and... \;\; \frac{\delta^2 F(S_t)}{\delta S_t^2} = -\frac{1}{S_t^2} \tag{8}$$

Substituting the derivatives of $F(S_t)$ as defined by Equation (8) into Equation (7) the equation for the change in the log of stock price becomes...

$$\delta F(S_t) = \frac{1}{S_t}\delta S_t - \frac{1}{2}\frac{1}{S_t^2}\delta S_t^2 \tag{9}$$

We will now make substitutions for δS_t and δS_t^2 in Equation (9) above. Given that δS_t is defined by Equation (6) we still need an equation for δS_t^2. Using Equation (6) above the equation for δS_t^2 is...

$$\delta S_t^2 = (S_t\,\mu\,\delta t)^2 = S_t^2\,\mu^2\,\delta t^2 = 0 \tag{10}$$

Note that in Equation (10) above $\delta S_t^2 = 0$ because $\delta t^2 = 0$. The value of the square of an infinitesimally small time period is zero. After substituting Equations (6) and (10) into Equation (9) the equation for the change in the log of stock price becomes...

$$\delta F(S_t) = \frac{1}{S_t}\Bigg(S_t\,\mu\,\delta t\Bigg) - \frac{1}{2}\frac{1}{S_t^2}\Bigg(0\Bigg) = \mu\,\delta t \tag{11}$$

Using Equation (11) above the equation for the log of stock price at any time t as a function of the log of stock price at time zero is...

$$F(S_t) = F(S_0) + \int_0^t \delta F(S_u) = F(S_0) + \int_0^t \mu\,\delta u = F(S_0) + \mu\,t \tag{12}$$

Given that stock price is the exponential of the log of stock price as defined by Equation (12) the equation for stock price at any time t as a function of stock price at time zero is...

$$S_t = S_0\,exp(\mu\,t) \tag{13}$$

By taking the first derivative of Equation (13) we get the ODE in Equation (6) above thus proving that Equation (13) is the solution to the ODE as defined by Equation (6). The proof is...

$$\frac{\delta S_t}{\delta t} = \frac{\delta S_0 \, e^{\mu t}}{\delta t} = \mu \, S_0 \, e^{\mu t} \;\; ...such\, that... \;\; \delta S_t = \mu \, S_0 \, e^{\mu t} \delta t = S_t \, \mu \, \delta t \tag{14}$$

How would we go about solving the problem above? To solve the problem in a continuous time framework we first convert the monthly compounded return to a continuously compounded return and then use continuous time Equation (13) to solve the problem. If 6% is the monthly compounded discrete time return and k is the continuous time equivalent return then the equation that converts the monthly compounded return into a continuously compouned return is...

$$\begin{aligned} e^k &= \left(1 + \frac{0.06}{12}\right)^{12} \\ k &= \ln \left\{ \left(1 + \frac{0.06}{12}\right)^{12} \right\} \\ k &= 0.05985 \end{aligned} \tag{15}$$

Stock price at the end of six months using Equation (13) is...

$$S_{0.50} = \$100.00 \times exp((0.05985)(0.50)) = \$103.04 \tag{16}$$

The answer to the problem assuming that the stock is risk-free is...

1) Stock price at the end of month six is \$103.04
2) Probability that the stock price will be greater than \$125 is zero because variance is zero

Introducing Risk Into The Equation

To incorporate risk into the ODE as defined by Equation (6) above we will add an innovation term such that the ordinary differential equation, which is deterministic in that stock price at some future time $t > 0$ is known with certainty at time zero, becomes a stochastic differential equation (SDE), which is a differential equation with a random component such that from the vantage point of time zero we don't know stock price at time $t > 0$ but we do know the possible paths that stock price may take and the attendant probabilities. The ODE in Equation (6) rewritten as an SDE is...

$$\delta S_t = S_t \, \mu \, \delta t + S_t \, \sigma \, \delta W_t \tag{17}$$

The δW_t in Equation (17) is the change in an underlying Brownian motion and is the source of randomness (i.e. uncertainty a.k.a. risk). Remember from Part II that the limiting distribution of a scaled symmetric random walk as the time step goes to zero is a Brownian motion with mean zero and variance t. The value of a scaled symmetric random walk at any time t can either increase or decrease by $\frac{1}{\sqrt{n}}$ over the time interval $[t, \, t + \frac{1}{n}]$.

We will define δt to be an infinitesimally small time period of near-zero length. Whereas δt is not insignificant the square of δt is. We therefore make the following definitions...

$$\lim_{n \to \infty} \frac{1}{n} = \delta t \;\; ...and... \;\; \lim_{n \to \infty} \left(\frac{1}{n}\right)^2 = \delta t^2 = 0 \tag{18}$$

If we define X_t to be the value of a scaled symmetric random walk at time t then the square of the increment in the Brownian motion over the time interval $[t, \, t + \frac{1}{n}]$ as n goes to infinity is...

$$\delta W_t^2 = \lim_{n \to \infty} \left(X_{t+\frac{1}{n}} - X_t\right)^2 = \lim_{n \to \infty} \left(\pm \frac{1}{\sqrt{n}}\right)^2 = \lim_{n \to \infty} \frac{1}{n} = \delta t \tag{19}$$

The square of the change in time over the time interval $[t, \, t + \frac{1}{n}]$ as n goes to infinity is...

$$\delta t^2 = \lim_{n \to \infty} \left(\left[t + \frac{1}{n}\right] - t\right)^2 = \lim_{n \to \infty} \left(\frac{1}{n}\right)^2 = 0 \tag{20}$$

The cross product of the change in the Brownian motion W_t and the change in time as n goes to infinity is...

$$\delta X_t \delta t = \lim_{n\to\infty} \left(X_{t+\frac{1}{n}} - X_t \right) \left(\left[t + \frac{1}{n} \right] - t \right) = \lim_{n\to\infty} \left(\pm \frac{1}{\sqrt{n}} \right) \left(\frac{1}{n} \right) = 0 \tag{21}$$

Using Equations (19), (20) and (21) above we will make the following definitions...

$$\delta t^2 = 0 \;\; ...and... \;\; \delta W_t \delta t = 0 \;\; ...and... \;\; \delta W_t^2 = \delta t \tag{22}$$

Risky Asset Prices in Continuous Time

We now have the tools to develop an equation for the evolution of stock price in continuous time where stock price is a function of a deterministic expected return and an unexpected innovation, which is a function of the underlying Brownian motion. Recall from Equation (9) above that given the function $F(S_t)$, which is the log of stock price, the change in $F(S_t)$ over the time period δt, which is a time period of infinitesimal length, can be described via the following second-order Taylor Series Expansion...

$$\delta F(S_t) = \frac{1}{S_t} \delta S_t - \frac{1}{2} \frac{1}{S_t^2} \delta S_t^2 \tag{23}$$

Note that when using the ODE from Equation (6) the second derivative in the equation above is zero, which is to be expected for equations that are not stochastic. Also recall from Equation (17) that the SDE for a risky asset such as our stock is...

$$\delta S_t = S_t \, \mu \, \delta t + S_t \, \sigma \, \delta W_t \tag{24}$$

Using the definitions from Equation (22) above the square of Equation (24) is...

$$\begin{aligned} \delta S_t^2 &= (S_t \, \mu \, \delta t + S_t \, \sigma \, \delta W_t)^2 \\ &= S_t^2 \, \mu^2 \, \delta t^2 + 2 \, S_t^2 \, \mu \, \sigma \, \delta W_t \, \delta t + S_t^2 \sigma^2 \delta W_t^2 \\ &= S_t^2 \sigma^2 \delta t \end{aligned} \tag{25}$$

After substituting Equations (24) and (25) into Equation (23) the equation for the change in the log of stock price over the time interval δt becomes...

$$\begin{aligned} \delta F(S_t) &= \frac{1}{S_t} \left\{ S_t \, \mu \, \delta t + S_t \, \sigma \, \delta W_t \right\} - \frac{1}{2} \frac{1}{S_t^2} \left\{ S_t^2 \sigma^2 \delta t \right\} \\ &= \mu \, \delta t + \sigma \, \delta W_t - \frac{1}{2} \sigma^2 \delta t \\ &= \left(\mu - \frac{1}{2} \sigma^2 \right) \delta t + \sigma \, \delta W_t \end{aligned} \tag{26}$$

Using Equation (26) above and noting that $W_0 = 0$ the equation for the log of stock price at any time t as a function of the log of stock price at time zero is...

$$\begin{aligned} F(S_t) &= F(S_0) + \int_0^t \delta F(S_u) \\ &= F(S_0) + \int_0^t \left(\mu - \frac{1}{2} \sigma^2 \right) \delta u + \int_0^t \sigma \, \delta W_u \\ &= F(S_0) + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma \, W_t \end{aligned} \tag{27}$$

Given that stock price is the exponential of the log of stock price as defined by Equation (27) the equation for stock price at any time t as a function of stock price at time zero is...

$$S_t = S_0 \, exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma \, W_t \right\} \;\; \text{...where...} \;\; W_t \sim N \left[0, t \right] \tag{28}$$

We can normalize Equation (28) such that the equation for stock price at any time t as a function of stock price at time zero becomes...

$$S_t = S_0 \, exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma \, \sqrt{t} \, Z \right\} \;\; \text{...where...} \;\; Z \sim N \left[0, \sigma^2 t \right] \tag{29}$$

Expected Asset Price

We will make the following definitions...

$$m=\left(\mu-\frac{1}{2}\sigma^2\right)t \text{ ...and... } v=\sigma^2\,t \tag{30}$$

Using the definitions in Equation (30) above we can rewrite Equation (29), which is the equation for stock price at any time t as a function of stock price at time zero, as...

$$S_t=S_0\exp\left\{\theta\right\} \text{ ...where... } \theta\sim N\left[m,v\right] \tag{31}$$

Using Equation (31) above the equation for expected stock price at time t is...

$$\mathbb{E}\left[S_t\right]=\mathbb{E}\left[S_0\exp\left\{\theta\right\}\right]=\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{2\,\pi\,v}}\exp\left\{-\frac{1}{2\,v}\left(\theta-m\right)^2\right\}S_0\exp\left\{\theta\right\}\delta\,\theta \tag{32}$$

Since the exponential of a normally-distributed random variate (θ) is a log-normally distributed random variate it can be shown that after solving the integral in Equation (32) above the equation for expected stock price at time t becomes...

$$\mathbb{E}\left[S_t\right]=S_0\exp\left\{m+\frac{1}{2}\,v\right\} \tag{33}$$

After substituting Equation (30) into Equation (33) the equation for expected stock price at time t becomes...

$$\mathbb{E}\left[S_t\right]=S_0\exp\left\{\left(\mu-\frac{1}{2}\,\sigma^2\right)t+\frac{1}{2}\,\sigma^2\,t\right\}=S_0\exp\left\{\mu\,t\right\} \tag{34}$$

The Answer To Our Hypothetical Problem

Since our stock is no longer risk-free (i.e. variance is greater than zero) the stock will earn the risky asset rate of return and therefore we need to convert the monthly compounded return on the risky asset to a continuously compounded return as we did in Equation (15) above. The continuous time risky asset rate of return is...

$$\begin{aligned}e^k&=\left(1+\frac{0.18}{12}\right)^{12}\\k&=\ln\left\{\left(1+\frac{0.18}{12}\right)^{12}\right\}\\k&=0.17866\end{aligned} \tag{35}$$

Using the equation for expected stock price as defined by Equation (34) above, we will drop in the parameters for our stock and solve for expected stock price at the end of month six, which is...

$$\mathbb{E}\left[S_{0.50}\right]=\$100.00\times\exp\left((0.17866)(0.50)\right)=\$109.34 \tag{36}$$

We will calculate the probability that stock price will be greater than \$125 by first solving for the value of the Brownian motion such that stock price is equal to \$125, which is...

$$\begin{aligned}S_t&=S_0\exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma\,W_t\right\}\\125.00&=100.00\times\exp\left\{\left(0.17866-\frac{1}{2}\times0.30^2\right)\times0.50+0.30\times W_t\right\}\\\frac{125.00}{100.00}&=\exp\left\{0.06685+0.30\times W_t\right\}\\W_t&=\frac{\ln(\frac{125.00}{100.00})-0.06685}{0.30}\\W_t&=0.521\end{aligned} \tag{37}$$

Given that the Brownian motion W_t has mean zero and variance t the probability of getting a value for the Brownian motion greater than 0.521 (and hence a stock price greater than \$125.00) is...

$$Prob[W_t > 0.521] = 1 - NORMDIST(0.521, 0, SQRT(0.50), TRUE) = 0.2306 \tag{38}$$

Note that we used the Excel function NormDist.

Answer: The probability of stock price being greater than \$125 in six months is approximately 23%.